One-DOF Rigid Foldable Structures from Space Curves

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Summary

We show a novel design method of one-DOF deployable mechanism based on a space curve, through creating a curved folding and discretizing the folding into rigid origami. By interpreting constant angle curved folding as a flat-foldable quadrilateral mesh origami, we design novel irregular tessellated, cylindrical, and cellular flatly collapsible structures, whose behavior is easily controlled by space curves.

Keywords: *deployable structure; structural morphology; origami, rigid foldable structure; curved folding.*

1. Introduction

Rigid origami structures, i.e., panels and hinges polyhedral surface mechanisms, can be used for designing deployable structures such as retractable roof, temporary shelters, space structures, and manipulators, or ordinary building elements such as solar shading, doors, and kinetic openings. Our objective is to morphologically investigate such structures to add more design freedom. In this context, quadrilateral panel structures is a powerful tool for designing rigid foldable structures since this yields one-DOF synchronized and robust mechanisms resulting from redundant mechanical constraints. This implies that the whole structure easily transforms by asserting force to a part of the structure. Examples of such mechanism are Miura-ori [1], Hoberman's designs [2], and rigid-foldable cylindrical polyhedra by Miura and Tachi [3]. Author's recent studies generalize theses structures to achieve further design freedom through perturbation-based computational methods to obtain variations of Miura-ori-like structures [4][5]. However, since these methods use continuous transformation from one pattern to another for the design, the achievable forms are restricted by the initial patterns. Finding the initial figures with controlled global shape and behavior is still an important yet unsolved problem.

In order to find such globally controllable form of one-DOF rigid origami structure, we propose a modeling method based on discretizing curved folding which can be constructed from a free single space curve.

The following shows the contribution of this work.

1. We show the geometry of discretized version of the curved folding constructed from a space curve.
2. We interpret the discretized version of the curved folding as a one-DOF mechanism and propose a novel method to construct a foldable structure that follows a given space curve or polyline.
3. We show that the method can be extended to tessellated, cylindrical, and cellular structures because of the flat-foldability originating from constant-angle folding.

We first start from the continuous curved folding, its construction from a space curve, and then discuss the discretized version. Next, we extend the method into tessellated, cylindrical, and cellular structures.

2. Curved Folding

There have been geometrical analysis and design proposals of curved folding. One of the early examples are studies by Huffman [6] and Resch [7]. A curved folded surface comprises uncreased (G^1 continuous) developable patches and the creases (curve on which surface is G^1 discontinuous) between them. An uncreased developable patch is represented as a collection of ruled surfaces in which the rulings have non-twisting nature.

2.1 Curved Folding from a Space Curve

It is known that a "nice" generic space curve, i.e., a ribbon curve, is a geodesic line of a developable surface. Such a curve can be assigned with local coordinates called Frenet–Serret frame. From a differential geometric analysis, it is known that such a curve can be interpreted as the folded curved crease of a developable surface as shown by computer graphics example created by Resch [7]. We can obtain different curved folded surfaces from one space curve by changing the folding angle. Fixing the folding angle function along the curve denoted by $2\alpha(t)$ (where t parameterizes the space curve by its arc length) uniquely determines the folded surface around the curve. The relationship between the folding angle and the ruling configuration has been investigated in Fuchs and Tabachnikov [8]. Here is a brief summary of their results with a more common notation.
First, we parameterize a given curve as $\mathbf{x}(t)$. We assume there exists a Frenet-Serret frame comprising three orthonormal vectors: tangent vector $\mathbf{T}(t)$, normal vector $\mathbf{N}(t)$, and bi-normal vector $\mathbf{B}(t)$. The curvature $\kappa(t)$ and torsion $\tau(t)$ relate to the frame in the following well-known formula:

$$\begin{bmatrix} \dot{\mathbf{T}}(t) \\ \dot{\mathbf{N}}(t) \\ \dot{\mathbf{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{bmatrix} \qquad (1)$$

The surface is represented by the crease and ruling pattern in 2D and the half of the folding angle $\alpha(t)$ at the crease. The local configuration of the former is characterized by curvature of the curve in 2D $\kappa_{2D}(t)$ and the ruling angles $\beta_L(t)$ and $\beta_R(t)$ defined as the angles between the rulings and the tangent vector at the point (Figure 1). The left and right tangent planes of the surface at t form angle $\alpha(t)$ with the osculating plane, i.e., a plane defined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$, and these variables are related as follows.

$$\kappa_{2D}(t) = \kappa(t)\cos\alpha(t) \qquad (2)$$

$$\cot\beta_L(t) = \frac{\alpha'(t) - \tau(t)}{\kappa_{2D}(t)\tan\alpha(t)} = \frac{\alpha'(t) - \tau(t)}{\kappa(t)\sin\alpha(t)} \qquad (3)$$

$$\cot\beta_R(t) = \frac{-\alpha'(t) - \tau(t)}{\kappa_{2D}(t)\tan\alpha(t)} = \frac{-\alpha'(t) - \tau(t)}{\kappa(t)\sin\alpha(t)} \qquad (4)$$

Equation (2) indicates that the ratio between curvatures in 3D and in 2D is high if the folding angle is big, and equations (3) and (4) imply that the torsion and the change in the folding angle affect the rulings configuration. We can use these equations to build up the curve-folded surface, crease pattern, and rulings from a space curve and the folding angle function $\alpha(t)$.

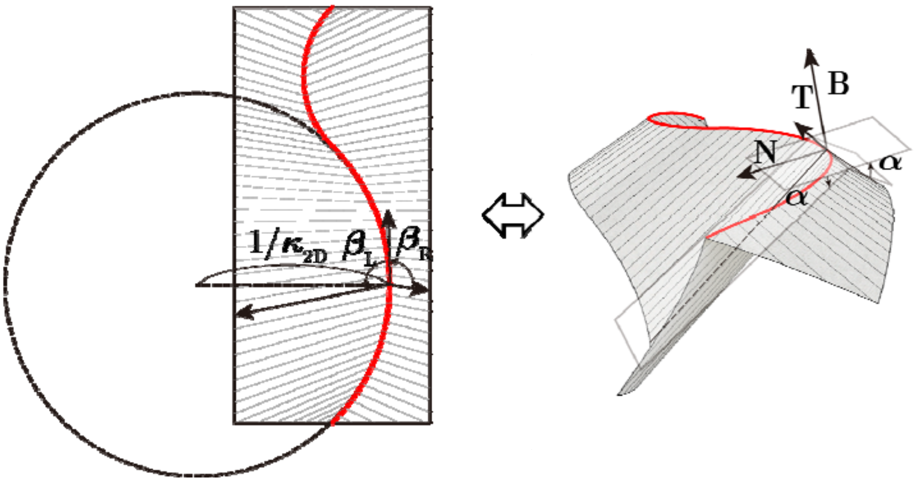

Fig.1 : Parameter of curved folding.

2.2 Constant Angle Curved Folding

Here we focus on the case of the constant-angle folding, where $\alpha(t) = \alpha = \text{const.}$. In this case, Equations (3) and (4) are combined into

$$\cot \beta_L(t) = \cot \beta_R(t) = \frac{-\tau(t)}{\kappa_{2D}(t)\tan\alpha} = \frac{-\tau(t)}{\kappa(t)\sin\alpha} . (5)$$

Therefore, in the crease pattern, the left and right ruling lines emanate at the same angle from the tangent vector of the curve, meaning that the rulings are drawn such that they reflect at the curve.

2.3 Ribbon

We can further build the surface from a given space curve (with finite $\kappa(t)$) and the folding function $\alpha(t) = \frac{\pi}{2}$ at every point, then the two developable surfaces lie onto each other. Then $\kappa_{2D}(t) = 0$, indicating that the curve is a straight line in 2D, i.e., a geodesic line of the surface. This is the shape of a straight strip of paper (or a ribbon) curved along the given space curve. The surface is locally on the plane defined by $\mathbf{B}(t)$ and $\mathbf{T}(t)$.

3. Discrete Curved Folding

Developable surfaces and curved folding surface can be represented computationally by discretely sampling the rulings as a planar-quadrilateral mesh as shown in Kergosien et al. [9], Bo and Wang [10], and Kilian et al. [11]. Here, the creases and rulings are discretized as the edges of the polyhedral mesh, and the non-twisting nature of the rulings is represented by the planarity of each quadrilateral facet. Here we show the process of constructing a discretized version of curved folding from a discretized space curve i.e., polyline.

3.1 Discrete Ribbon

We can construct a discrete ribbon from a given polyline so that the center line of the ribbon is the given polyline. The construction is performed sequentially starting from one edge and propagate from there. Suppose that an edge E_i is assigned with a frame comprising $\mathbf{T}_i$, $\mathbf{N}_i$, and $\mathbf{B}_i$ so that the incident quadrangle patch Q_i is perpendicular to $\mathbf{N}_i$. Our objective is to derive the rulings vector

$\mathbf{r}_{i+1}$ emanating from the incident point of edges i and $i+1$ from $\mathbf{T}_i$, $\mathbf{N}_i$, $\mathbf{B}_i$ and $\mathbf{T}_{i+1}$ (note that $\mathbf{T}_0$, ..., $\mathbf{T}_n$ are given as the directions of the edges) (see Fig. 2). We can identify $\mathbf{r}_{i+1}$ by three constraints: 1) since $\mathbf{r}_{i+1}$ is the intersection of the Q_i and Q_{i+1}, $\mathbf{r}_{i+1}$ is perpendicular to $\mathbf{N}_i$; 2) since polyline E_iE_{i+1} is a geodesic line of the quadrangle strip, the angle between $\mathbf{T}_i$ and $\mathbf{r}_{i+1}$ equals to that of $\mathbf{T}_{i+1}$ and $\mathbf{r}_{i+1}$; 3) the distance from the end point of $\mathbf{r}_{i+1}$ from the edge should be half of the width of the strip $w/2$. Therefore, the ruling vector is given by solving the following matrix equation.

$$\begin{bmatrix} \mathbf{N}_i \\ (\mathbf{T}_{i+1} - \mathbf{T}_i)^T \\ \mathbf{B}_i \end{bmatrix} \mathbf{r}_{i+1} = \begin{bmatrix} 0 \\ 0 \\ w/2 \end{bmatrix} \qquad (6)$$

Once we define $\mathbf{r}_{i+1}$, then the next frame is calculated by $\mathbf{N}_{i+1} = \mathbf{r}_{i+1} \times \mathbf{T}_{i+1}$ and $\mathbf{B}_{i+1} = \mathbf{T}_{i+1} \times \mathbf{N}_{i+1}$. In the discrete case, the frame of the initial edge is freely determined. However, in order to be consistent with a smooth curve case with infinite number of vertices, we approximate the initial frame using $\mathbf{T}_{i-1}$, $\mathbf{T}_i$, and $\mathbf{T}_{i+1}$.

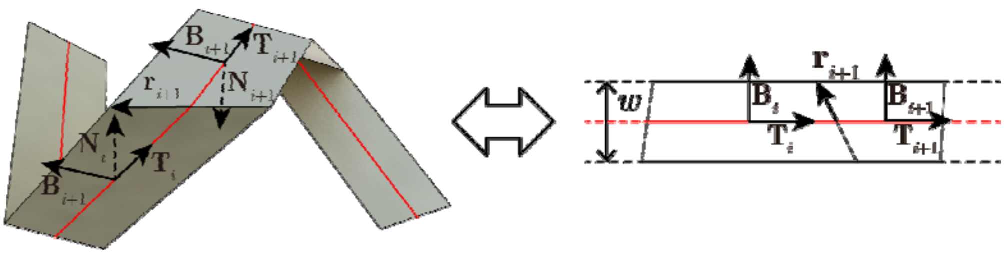

Fig. 2: Discrete Ribbon.

3.2 Constant Angle Discrete Curved Folding

Once we assign consistent frame for every edge and obtain the ribbon, we can construct a discrete curved folding by replacing the strip with V-shaped ridge as shown in Figure 3. The process is as follows.

1. Construct a pair of symmetric half-planes through each edge i to form angle $\frac{\pi}{2} - \alpha$ with plane Q_i.
2. Trim corresponding adjacent planes at the intersection line and join them to form a continuous ridge.

This can be processed in the same manner as in constructing ruling vectors of a ribbon by just rotating each frame before the calculation. This construction procedure yields a developable and flat-foldable pattern for a certain range of α. Note that the process relies on the special condition under constant-angle folding and does not apply to the case where each edge has different folding angle.

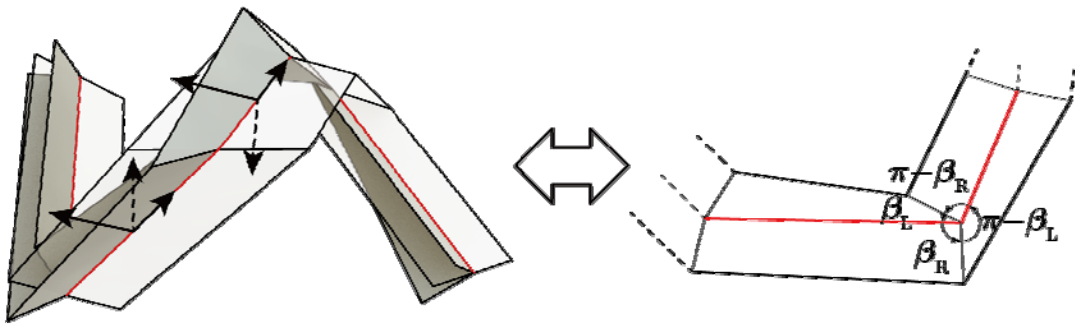

Fig. 3: Discrete curved folding with constant folding angle..Note that every vertex is flat-foldable in the isometric sense, i.e., Kawasaki's theorem is satisfied (alternating sum of incident angles is 0).

4. Rigid Origami Structures

Here, we interpret such a polyhedral mesh as a rigid origami structure and propose a novel method for designing rigid-foldable structure from a space curve. Even though a physical curved folding of real material is floppy and flexible, the polyhedral structure constructed from a curved folding has less flexibility than the original material. This is because the change in the position and orientation of a ruling is not allowed in the rigid origami structures.

The mechanism of the structure is characterized by the number of degrees of freedom (DOF). This is generically calculated by the number of variables (fold angle of each edge) and constraints (each vertex producing three constraints). Therefore an interior vertex composed of four edges produces a one-DOF mechanism, i.e., one of the folding angles determines the entire configuration of the four foldlines. Therefore, a planar quadrangle mesh composed of degree-4 vertices yields the following structure.

1. A $2\times n$ quadrangle array (a single crease curved folding) yields one-DOF mechanism (Fig. 4).
2. A $m\times n$ (where $m>2$) quadrangle array (a pattern with more than two creases) yields over-constrained static structure or a redundant one-DOF mechanism because fold angles are multiply defined (Fig. 4).

The necessary and sufficient condition of the pattern for having a one-DOF mechanism is not perfectly investigated. However, we can use special cases that the planar quad mesh produces redundant constraints: when the crease pattern is flat-foldable and there is at least one 3D state [3]; and when the panels are constructed from parallelograms to form identity constraints around each facet. Here, we will start from a single crease curved folding and then extend to general patterns using similarity.

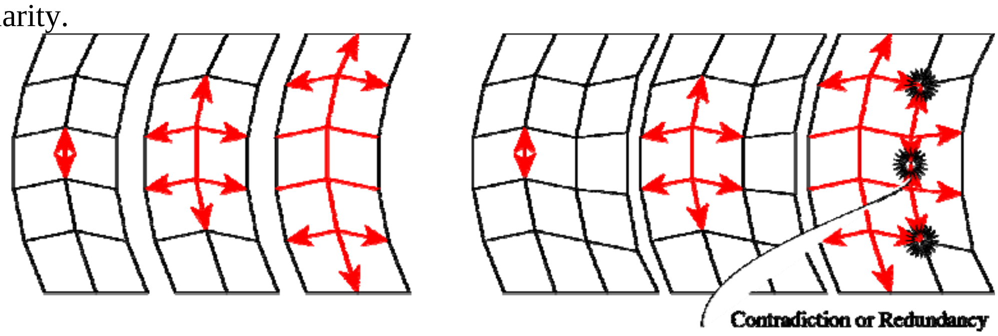

Fig. 4: One-DOF foldable mechanism (left) vs. over constrained mechanism (right).

4.1 Single Curve

As described before, a single curved folding that can be constructed from a space polyline can be discretized to form a one-DOF mechanism (Fig. 5). We can observe that the curvature and torsion of the form increase by folding. This can be understood as the change in parameters of Equation (2) and (5) when ruling and crease pattern parameters β and κ_{2D} are fixed.

Since the curvature only increases as the surface folds, structure from a smooth curve generally tries to roll up as shown in Fig. 5 left. This often causes collision between facets for a general curve, and the folding cannot go up to actual flat-folding even though flat-foldability of the pattern in isometric sense is satisfied.

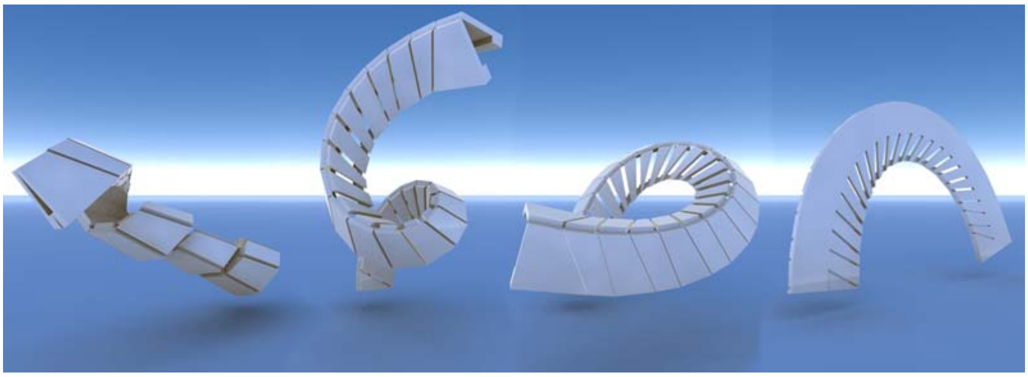

Fig. 5: Folding sequence of the structure from single curve. Solid model based on thick panel construction technique [5].

One variational form that allows actual flat-folding and thus compact packaging is to use a zig-zagged line (alternating curvature) instead of polyline that directly follows a given curve (Fig. 6). The zig-zagged line is constructed from a directly sampled polyline by 1) constructing the frames for the polyline and 2) moving vertices alternately along the curvature vector.

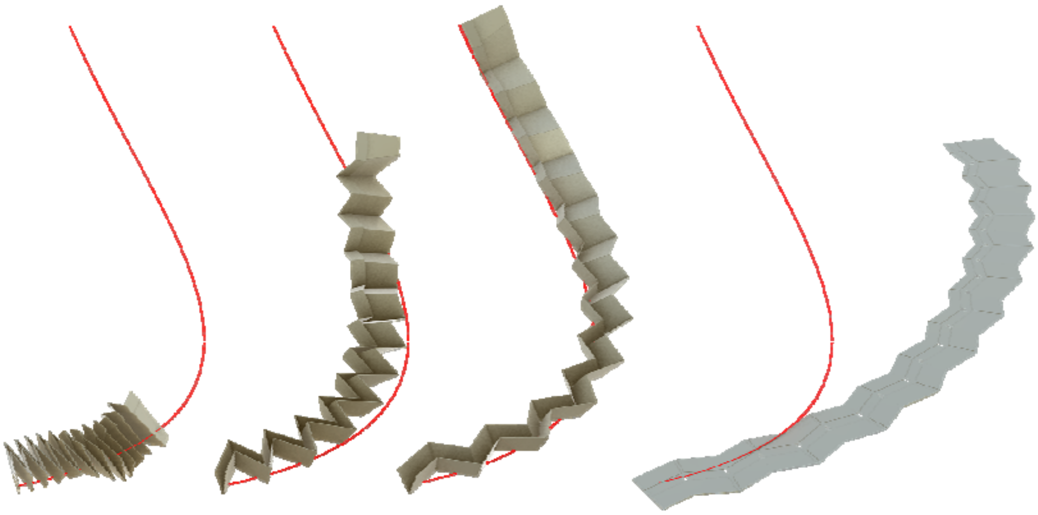

Fig. 7: Folding sequence of the zig-zagged structure. The folded form approximates the original space curve drawn in red.

4.2 Planar Tessellation

We can extend the structure from $2\times n$ to $m\times n$ (where $m>2$) by constructing patterns that avoids contradiction between the mechanisms of adjacent vertices. Such a pattern can be constructed by using parallel lines to copy and duplicate adjacent vertex structures. Fig. 8 shows how the array can be extended in width direction. 1) First, trim the developable surface so that every perimeter edge is parallel to adjacent crease edge. 2) Then, construct quadrangle strips so that every other strip has the parallel sequence of rulings vectors. Because of the similarity of vertices, we can assign consistent folding angle function to each crease. The resulting structure is a generalized Miura-ori surface that are bent and twisted along the curve (Fig. 9).

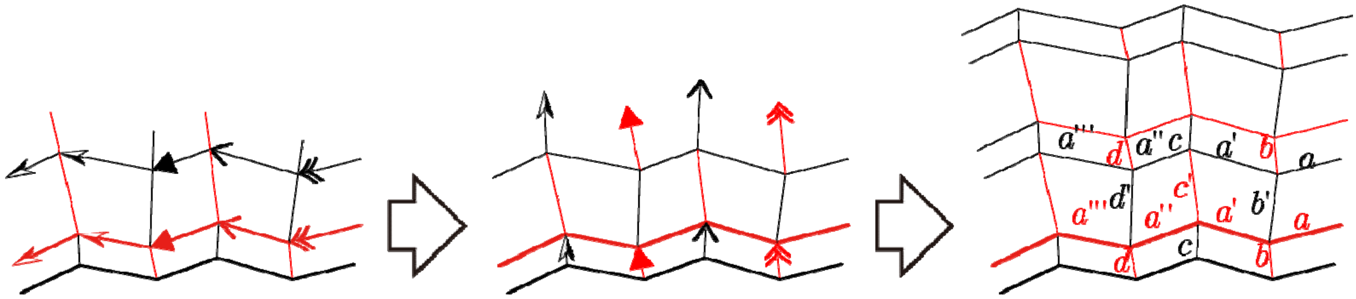

Fig. 8: Construction of planar tessellation with rigid-foldability.

4.3 Cylindrical and Cellular Structures

Our method can also extend to cylindrical and cellular structures since the folded strips satisfy flat-foldability condition necessary for the construction of cylindrical structures [5]. A cylinder can be constructed similarly to the planar tessellation using parallel lines. We chose the inverse direction for attaching a strip then we can loop back to the original strip. Since the section is a parallelogram, we can obtain a transformable cylindrical structure. Additionally, if we construct two surfaces in both directions at each time, we can obtain a cellular structure that fills three-dimensional volume (Fig. 10). Fig. 11 shows an example design of deployable cylindrical structure.

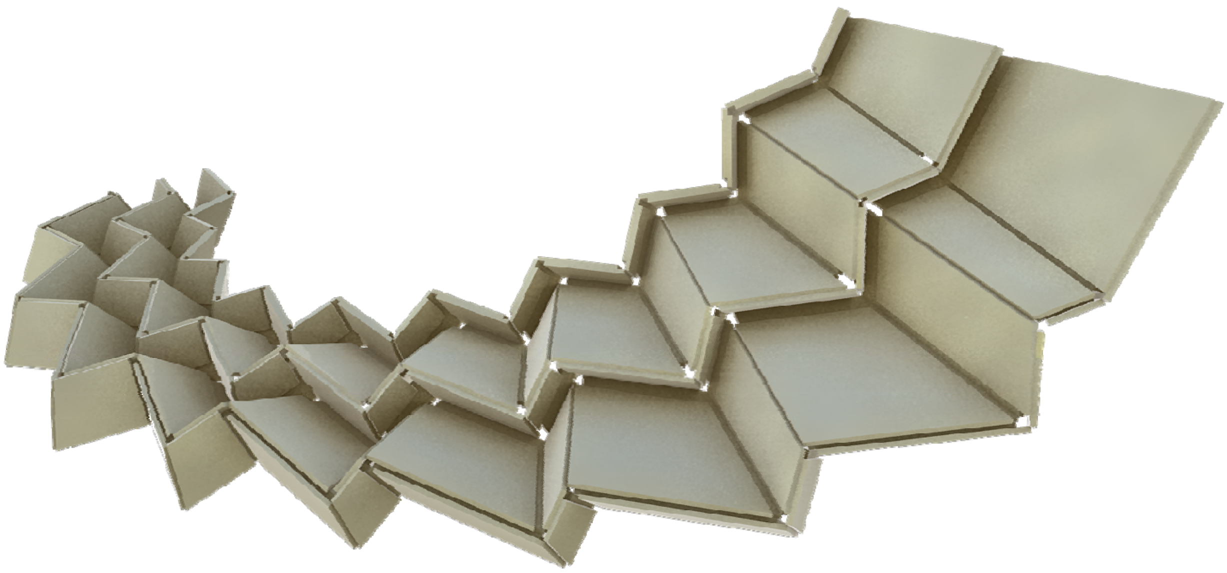

Fig. 9: Tessellated Rigid-Foldable Structure.

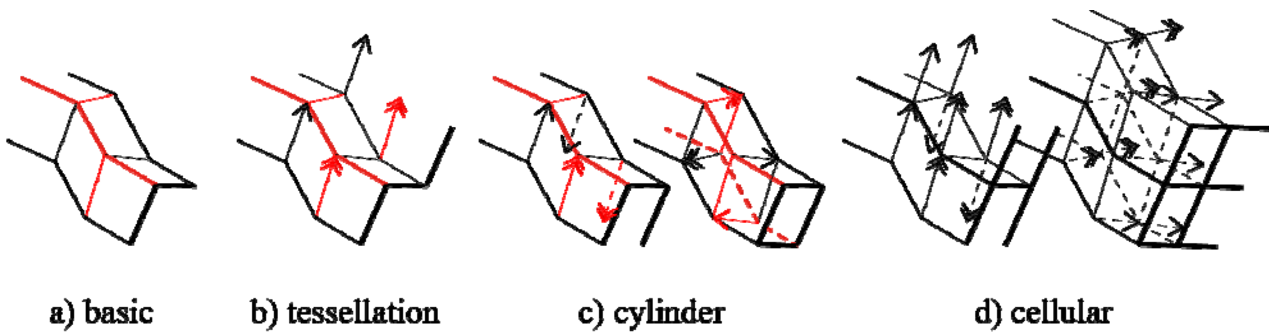

Fig. 10: Constructing Cylindrical and Cellular Structures.

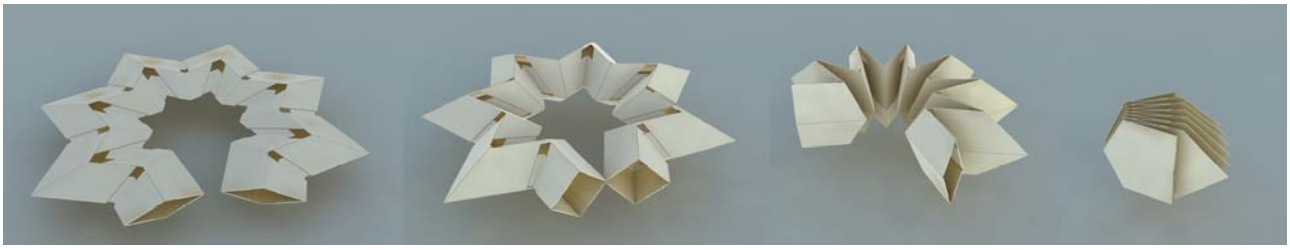

Fig. 11: Example Cylindrical Structure.

5. Conclusion

We have shown a family of design methods of deployable linkages with a one-DOF motion constructed from single space curve. We enabled a compactly packageable geometric structure, by using the constant angle curved folding that can be interpreted as a flat-foldable pattern. The basic method can extend to corrugated structures based on zigzagged space curves, planar tessellation exhibiting Miura-ori like form, and cylindrical and cellular collapsible structures whose shapes are freely controlled by space curves.

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